On the Initial Value Problem and Exact Solutions for the Higher-order Broer-Kaup Equations

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The Backlund transformation and exact solutions, especially the closed form of the solution for the initial value problem of the higher order Broer-Kaup (BK) systems in (1+1) and (2+1) dimensions, are obtained by using the extended homogeneous balance method. The method used here is simple and can be generalized to deal with other classes of nonlinear equations.

Key words: Higher-order Broer-Kaup Systems in (1+1) and (2+1) Dimensions; Extended Homogeneous Balance Method; Backlund Transformation; Exact Solutions; Initial Value Problem.

1. Introduction

Nonlinear differential equations are known to describe a wide variety of phenomena not only in physics, in which applications extend over magnetofluid dynamics, water surface gravity waves, electromagnetic radiation reactions, and ion acoustic waves in plasmas, just to name a few, but also in biology and chemistry. It is one of the important tasks in the study of nonlinear differential equations to seek exact and explicit solutions. In the past several decades both mathematicians and physicists have made many attempts in this direction [1 – 11].

In the present paper we consider the higher-order Broer-Kaup (BK) systems in (1+1) dimensions,

$$H_t = -4(H_{xx} + H^3 + 6HG - 3HH_x)_x, \tag{1}$$

$$G_t = -4(G_{xx} + 3G_xH + 3H^2G + 3G^2)_x, \tag{2}$$

and (2+1) dimensions,

$$H_t = -4(H_{xx} + H^3 - 3HH_x + 3H\partial_z^{-1}G_x + 3\partial_z^{-1}(GH)_x)_x, \tag{3}$$

$$G_t = -4(G_{xx} + 3G_xH + 3H^2G + 3G\partial_z^{-1}G_x)_x, \tag{4}$$

where $\partial_z^{-1} = \int \mathrm{d}z$. These systems may be deduced from the Kadomtsev-Petviashvili equation by using symmetry constraints [12, 13]. By using the extended homogeneous balance method [14], the Backlund transformations for the BK systems in (1+1) and (2+1) dimensions are derived. Connections between the BK equations and the Korteweg-de Vries (KdV) equations are found, which are used to obtain three families of solutions for the BK equations. In these solutions a simple solitary wave solution and infinitely many rational function solutions, especially the closed form of the solution for the initial value problem of these systems, are included.

2. The Initial Value Problem and Exact Solutions of the Higher-order BK in (1+1) Dimensions

According to the extended homogeneous balance method, let

$$H(x,t) = f'(\varphi)\varphi_x + P(x,t), \tag{5}$$

$$G(x,t) = g^{(2)}(\varphi)\varphi_x^2 + g'(\varphi)\varphi_{xx} + Q(x,t), \tag{6}$$

where $P(x,t)$ and $Q(x,t)$ are known solutions of the system expressed by (1) and (2).

Substituting (5) and (6) into (1) and (2), we obtain

$$H_t + 4(H_{xx} + H^3 + 6HG - 3HH_x)_x =$$
$$(4f^{(4)} + 12f'^2 f'' + 24f''g'' + 24f'g^{(3)} - 12f^{(3)}f' - 12f''^2)\varphi_x^4 + \text{lower power terms of the derivatives of } \varphi(x,t) \text{ with respect to } x \text{ and } t, \tag{7}$$

and

$$G_t+4(G_{xx}+3G_xH+3H^2G+3G^2)_x=$$
$$(4g^{(5)}+12g^{(4)}f'+12g^{(3)}f''+24g''f''f'+12g^{(3)}f'^2+24g^{(3)}g'')\varphi_x^5+\text{lower power terms of the derivatives of } \varphi(x,t) \text{ with respect to } x \text{ and } t. \quad (8)$$

Setting the coefficients of φ_x^4 in (7) and φ_x^5 in (8) to zero yields an ordinary differential system for $f(\varphi)$ and $g(\varphi)$:

$$4f^{(4)}+12f'^2f''+24f''g''+24f'g^{(3)}-12f^{(3)}f'-12f''^2=0, \quad (9)$$

$$4g^{(5)}+12g^{(4)}f'+12g^{(3)}f''+24g''f''f'+12g^{(3)}f'^2+24g^{(3)}g''=0 \quad (10)$$

There exists a special solution for (9) and (10):

$$f(\varphi)=g(\varphi)=\ln(\varphi). \quad (11)$$

Thereby

$$f'^4=-\frac{1}{6}f^{(4)},\ f''^2=-\frac{1}{6}f^{(4)},\ f'f^{(3)}=-\frac{1}{3}f^{(4)},$$
$$f'^2f''=\frac{1}{6}f^{(4)},\ f'^3=\frac{1}{2}f^{(3)},\ f'f''=-\frac{1}{2}f^{(3)}, \quad (12)$$
$$f'^2=-f^{(2)}.$$

Using (11) and (12), the expressions (7) and (8) can be simplified as

$$H_t+4(H_{xx}+H^3+6HG-3HH_x)_x=(\varphi_x\varphi_t+4\varphi_x\varphi_{xxx}+12\varphi_x\varphi_{xx}P-12\varphi_x^2P_x+12\varphi_x^2P^2+24\varphi_x^2Q)f^{(2)}$$
$$+(\varphi_{xt}+4\varphi_{xxxx}+12\varphi_{xxx}P+12\varphi_{xx}P^2+24\varphi_xPP_x+24\varphi_{xx}Q+24\varphi_xQ_x-12\varphi_xP_{xx})f'$$
$$+P_t+4(P_{xx}+P^3+6PQ-3PP_x)_x=0, \quad (13)$$

$$G_t+4(G_{xx}+3G_xH+3H^2G+3G^2)_x=(\varphi_x^2\varphi_t+4\varphi_x^2\varphi_{xxx}+12\varphi_x^2\varphi_{xx}P+12\varphi_x^3P^2+12\varphi_x^3Q)f^{(3)}$$
$$+(\varphi_{xx}\varphi_t+2\varphi_x\varphi_{xt}+4\varphi_{xx}\varphi_{xxx}+8\varphi_x\varphi_{xxxx}+12\varphi_{xx}^2P+24\varphi_x\varphi_{xxx}P+12\varphi_x\varphi_{xx}P_x+24\varphi_x^2Q_x+24\varphi_x^2PP_x$$
$$+36\varphi_x\varphi_{xx}P^2+24\varphi_x^2PQ+48\varphi_x\varphi_{xx}Q)f''$$
$$+(\varphi_{xxt}+4\varphi_{xxxxx}+12\varphi_xQ_{xx}+12\varphi_{xxxx}P+12\varphi_{xxx}P_x+12\varphi_{xx}Q_x+24\varphi_{xx}PP_x+12\varphi_{xxx}P^2+24\varphi_xPQ_x$$
$$+24\varphi_{xx}PQ+24\varphi_xP_xQ+24\varphi_{xxx}Q+24\varphi_{xx}Q_x)f'$$
$$+Q_t+4(Q_{xx}+3Q_xP+3P^2Q+3Q^2)_x=0. \quad (14)$$

Setting the coefficients of $f^{(3)}$, f'', f' in (13) and (14) to zero yields a set of partial differential equations for $\varphi(x,t)$:

$$\varphi_x\varphi_t+4\varphi_x\varphi_{xxx}+12\varphi_x\varphi_{xx}P-12\varphi_x^2P_x+12\varphi_x^2P^2+24\varphi_x^2Q=0, \quad (15)$$

$$\varphi_{xt}+4\varphi_{xxxx}+12\varphi_{xxx}P+12\varphi_{xx}P^2+24\varphi_xPP_x+24\varphi_{xx}Q+24\varphi_xQ_x-12\varphi_xP_{xx}=0, \quad (16)$$

$$\varphi_x^2\varphi_t+4\varphi_x^2\varphi_{xxx}+12\varphi_x^2\varphi_{xx}P+12\varphi_x^3P^2+12\varphi_x^3Q=0, \quad (17)$$

$$\varphi_{xx}\varphi_t+2\varphi_x\varphi_{xt}+4\varphi_{xx}\varphi_{xxx}+8\varphi_x\varphi_{xxxx}+12\varphi_{xx}^2P+24\varphi_x\varphi_{xxx}P+12\varphi_x\varphi_{xx}P_x+24\varphi_x^2Q_x+24\varphi_x^2PP_x+36\varphi_x\varphi_{xx}P^2+24\varphi_x^2PQ+48\varphi_x\varphi_{xx}Q=0, \quad (18)$$

$$\varphi_{xxt}+4\varphi_{xxxxx}+12\varphi_xQ_{xx}+12\varphi_{xxxx}P+12\varphi_{xxx}P_x+12\varphi_{xx}Q_x+24\varphi_{xx}PP_x+12\varphi_{xxx}P^2+24\varphi_xPQ_x+24\varphi_{xx}PQ+24\varphi_xP_xQ+24\varphi_{xxx}Q+24\varphi_{xx}Q_x=0. \quad (19)$$

Analyzing (15–19), we find that when $\varphi_x\neq 0$, the independent equation in (15–19) reads

$$\varphi_t+12P^2\varphi_x+12(P\varphi_x)_x+4\varphi_{xxx}=0, \quad (20)$$

provided that

$$P_x=Q. \quad (21)$$

Substituting (11) into (5) and (6), we obtain a Backlund transformation of the BK equations in (1+1) dimensions

$$H(x,t)=\partial\ln\varphi/\partial x+P, \quad (22)$$

$$G(x,t)=\partial^2\ln\varphi/\partial x^2+Q, \quad (23)$$

where φ,P,Q satisfy (20) and (21),

$$Q_t+4(Q_{xx}+3Q_xP+3P^2Q+3Q^2)_x=0,$$
$$P_t+4(P_{xx}+P^3+6PQ-3PP_x)_x=0.$$

i) Putting $P_x = Q = 0$, $P = C_0$ = const, we obtain a transformation of the BK equations in (1+1) dimensions from (20–23).

$$H(x,t) = \partial \ln\varphi/\partial x + C_0, \tag{24}$$

$$G(x,t) = \partial^2 \ln\varphi/\partial x^2. \tag{25}$$

The above transformation is the transformation between (1), (2) and (20). By using this transformation, it is obvious that the higher order Broer-Kaup equations can be transformed into linear equations $\varphi_t + 12C_0^2\varphi_x + 12C_0\varphi_{xx} + 4\varphi_{xxx} = 0$.

ii) Putting $P = \varphi$, $Q = \varphi_x$, we can obtain another transformation of the BK equations in (1+1) dimensions from (20)-(23):

$$H(x,t) = \partial \ln\varphi/\partial x + \varphi, \tag{26}$$

$$G(x,t) = \partial^2 \ln\varphi/\partial x^2 + \varphi_x. \tag{27}$$

The transformation (26), (27) is the transformation between (1), (2) and $\varphi_t + 12\varphi^2\varphi_x + 12(\varphi\varphi_x)_x + 4\varphi_{xxx} = 0$, and it is an auto-Backlund transformation.

iii) From (20), (24) and (25), we can construct exact solutions of the BK equations in (1+1) dimensions:

It can be verified that there are the following solutions of (20):

$$\varphi_1(x,t) = 1 + A\exp(36C_0^3 t)\sin(\sqrt{3}C_0 x), \tag{28}$$

$$\varphi_2(x,t) = 1 + A\exp(36C_0^3 t)\cos(\sqrt{3}C_0 x), \tag{29}$$

$$\varphi_3(x,t) = 1 + \exp[\alpha x + (-12C_0^2\alpha - 12C_0\alpha^2 - 4\alpha^3)t + \gamma]. \tag{30}$$

Substituting (28–30) into (24) and (25), we obtain two groups of exact solutions of the system (1) and (2) as follows:

$$H_1(x,t) = \frac{\sqrt{3}C_0 A\exp(36C_0^3 t)\cos(\sqrt{3}C_0 x)}{1 + A\exp(36C_0^3 t)\sin(\sqrt{3}C_0 x)} + C_0, \tag{31}$$

$$G_1(x,t) = -3C_0^2 A\exp(36C_0^3 t) \cdot \frac{\sin(\sqrt{3}C_0 x) + A\exp(36C_0^3 t)}{[1 + A\exp(36C_0^3 t)\sin(\sqrt{3}C_0 x)]^2}, \tag{32}$$

$$H_2(x,t) = -\frac{\sqrt{3}C_0 A\exp(36C_0^3 t)\sin(\sqrt{3}C_0 x)}{1 + A\exp(36C_0^3 t)\cos(\sqrt{3}C_0 x)} + C_0, \tag{33}$$

$$G_2(x,t) = -3C_0^2 A\exp(36C_0^3 t) \cdot \frac{\cos(\sqrt{3}C_0 x) + A\exp(36C_0^3 t)}{[1 + A\exp(36C_0^3 t)\cos(\sqrt{3}C_0 x)]^2}, \tag{34}$$

and a solitary wave solution

$$H_3(x,t) = \frac{1}{2}\alpha\tanh\left\{\frac{1}{2}\left[\alpha x + (-12C_0^2\alpha - 12C_0\alpha^2 - 4\alpha^3)t + \gamma\right]\right\} + \frac{1}{2}\alpha + C_0, \tag{35}$$

$$G_3(x,t) = \frac{1}{4}\alpha^2\mathrm{sech}^2\left\{\frac{1}{2}\left[\alpha x + (-12C_0^2\alpha - 12C_0\alpha^2 - 4\alpha^3)t + \gamma\right]\right\}, \tag{36}$$

where A, C_0, α, and γ are arbitrary constants.

By using the separation of variables approach, we can obtain another set of real valued general solutions of (20) in the case $P_x = Q = 0$, $P = C_0$ = const:

$$\varphi_4(x,t) = e^{-4C_0^3\lambda t}\left\{C_1 e^{C_0[(\lambda+1)^{\frac{1}{3}}-1]x} + C_2 e^{-\frac{C_0}{2}[(\lambda+1)^{\frac{1}{3}}+2]x}\cos\left[\frac{\sqrt{3}}{2}(\lambda+1)^{\frac{1}{3}}C_0 x\right] + C_3 e^{\frac{-C_0}{2}[(\lambda+1)^{\frac{1}{3}}+2]x}\sin\left[\frac{\sqrt{3}}{2}(\lambda+1)^{\frac{1}{3}}C_0 x\right]\right\}, \tag{28'}$$

where C_0, C_1, C_2, C_3, and λ are arbitrary constants.

Substituting (28') into (24) and (25), we obtain exact solutions of the system (1) and (2).

To show the properties of the solutions obtained, we draw plots of their solutions $H_1(x,t)$, $G_1(x,t)$, $H_3(x,t)$, and $G_3(x,t)$ (see Figs. 1–4). The plots for the solutions $H_2(x,t)$, $G_2(x,t)$ are omitted since their shapes are very similar to those of the modulus of the solutions $H_1(x,t)$, $G_1(x,t)$.

It is important that in the case of $P_x = Q = 0$, $P = C_0$ = const, the solution of the initial value problem for the higher order BK system in (1+1) dimensions can be written in the closed form

$$H(x,t) = C_0 + \frac{\partial}{\partial x}\ln[\varphi(x,t)], \tag{37}$$

$$G(x,t) = \frac{\partial}{\partial x}H(x,t), \tag{38}$$

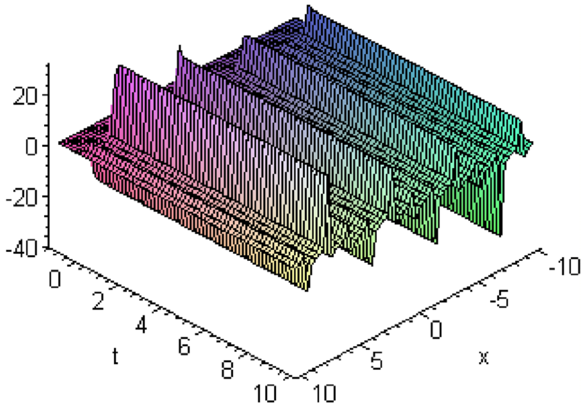

Fig. 1. The solution $H_1(x,t)$, where parameters are $C_0 = 1$, $A = 1$.

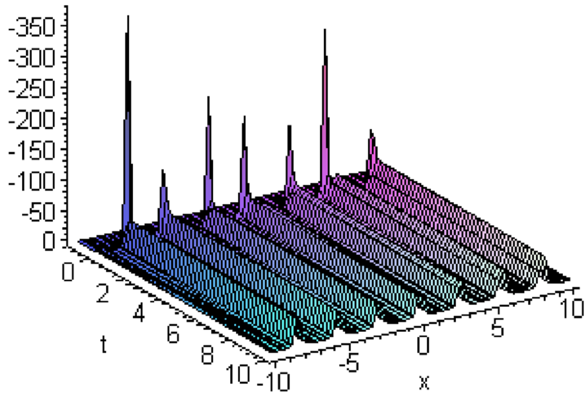

Fig. 2. The solution $G_1(x,t)$, where parameters are $C_0 = \frac{2}{3}$, $A = 1$.

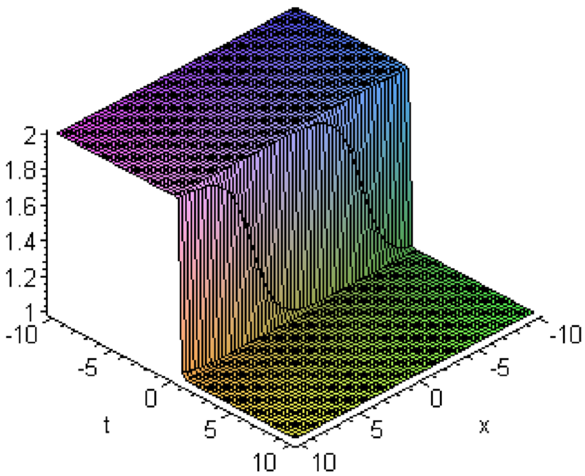

Fig. 3. The solution $H_3(x,t)$, where parameters are $C_0 = \alpha = \gamma = 1$.

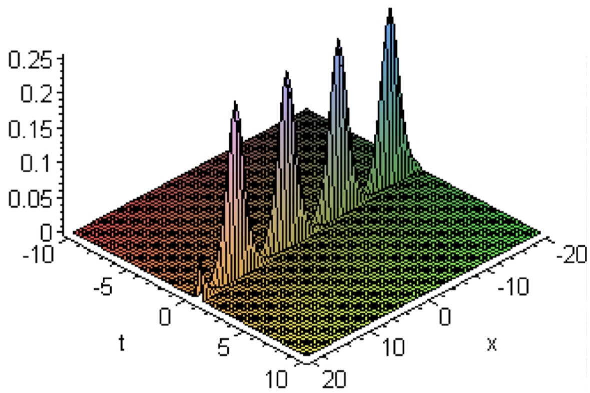

Fig. 4. The solution $G_3(x,t)$, where parameters are $C_0 = \alpha = \gamma = 1$.

where

$$\varphi(x,t) = \frac{\varphi(0,0)}{2\pi(12t)^{\frac{1}{3}}} \int_{-\infty}^{\infty} \exp\left[\int_0^{\zeta} H(x,0)\mathrm{d}x - C_0\zeta + C_0(x-\zeta) - 20C_0^3 t\right] Ai\left(\frac{24C_0^2 t - (x-\zeta)}{(12t)^{\frac{1}{3}}}\right)\mathrm{d}\zeta, \quad (39)$$

and $Ai(x)$ is the Airy function.

In fact, making a Fourier transformation of (20) with respect to the variable x, we obtain

$$\Psi_t(k,t) + 12C_0^2 ik\Psi(k,t) - 12C_0k^2\Psi(k,t) - 4ik^3\Psi(k,t) = 0, \quad (40)$$

where

$$\Psi(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x,t)e^{-ikx}\mathrm{d}x. \quad (41)$$

Integrating the ordinary differential equation (40) with respect to the variable t, we obtain

$$\Psi(k,t) = \Psi(k,0)\exp[(4ik^3 + 12C_0k^2 - 12C_0^2 ik)t], \quad (42)$$

where

$$\Psi(k,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x,0)e^{-ikx}\mathrm{d}x. \quad (43)$$

Making the inverse Fourier transformation of $\Psi(k,t)$

with respect to k, we obtain

$$\varphi(x,t) = \frac{1}{2\pi(12t)^{\frac{1}{3}}} \int_{-\infty}^{\infty} \varphi(\zeta,0)\exp[C_0(x-\zeta) - 20C_0^3 t] \cdot Ai\left(\frac{24C_0^2 t - (x-\zeta)}{(12t)^{\frac{1}{3}}}\right) \mathrm{d}\zeta. \quad (44)$$

Starting from (5), and using $P(x,t) = C_0$, we obtain

$$\varphi(\zeta,0) = \varphi(0,0)\exp\left[\int_0^{\zeta} H(x,0)\mathrm{d}x - C_0\zeta\right]. \quad (45)$$

Substituting $\varphi(\zeta,0)$ into (44), we complete the proof of (39). Because $\varphi(0,0)$ does not appear in the solution (37) and (38), the initial value problem of (1) and (2) is determined completely up to an arbitrary constant C_0.

iv) Make the ansatz $P_x = Q = -\frac{C}{(x+x_0)^2}$, $P = \frac{C}{x+x_0}$, $C = \frac{3\pm 1}{2}$, where x_0 is an arbitrary constant.

In order to solve (20), suppose that

$$\varphi(x,t) = \sum_{n=0}^{n=\infty} a_n(t)(x+x_0)^n \quad (46)$$

and substitute this $\varphi(x,t)$ in (46) into (20). This implies

$$\sum_{n=0}^{n=\infty} \Big[\frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + 12C^2 a_{n+3}(t)(n+3) + 12Ca_{n+3}(t) \cdot (n+3)(n+1) + 4a_{n+3}(t)(n+3)(n+2)(n+1)\Big] \cdot (x+x_0)^n = 0. \quad (47)$$

Setting the coefficient of $(x+x_0)^n$ for any n to zero yields a set of ordinary differential equations for the $a_n(t)$:

$$\frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + 12C^2 a_{n+3}(t)(n+3) + 12Ca_{n+3}(t)(n+3) \cdot (n+1) + 4a_{n+3}(t)(n+3)(n+2)(n+1) = 0. \quad (48)$$

In order to solve this set of ordinary differential equations, we must cut the series solution $\varphi(x,t)$ in (46) off at some n artificially. For example, if we let $a_n \equiv 0$ (for $n \geq N, N$ is an arbitrary constant), we may solve the system as follows

$$\frac{\mathrm{d}a_N(t)}{\mathrm{d}t} = 0, \quad (49)$$

$$\frac{\mathrm{d}a_{N-1}(t)}{\mathrm{d}t} = 0, \quad (50)$$

$$\frac{\mathrm{d}a_{N-2}(t)}{\mathrm{d}t} = 0, \quad (51)$$

$$\frac{\mathrm{d}a_{N-3}(t)}{\mathrm{d}t} + [12C^2 N + 12CN(N-2) + 4N(N-1)(N-2)]a_N(t) = 0, \quad (52)$$

$$\frac{\mathrm{d}a_{N-4}(t)}{\mathrm{d}t} + [12C^2(N-1) + 12C(N-1)(N-3) + 4(N-1)(N-2)(N-3)]a_{N-1}(t) = 0, \quad (53)$$

$$\frac{\mathrm{d}a_{N-5}(t)}{\mathrm{d}t} + [12C^2(N-2) + 12C(N-2)(N-4) + 4(N-2)(N-3)(N-4)]a_{N-2}(t) = 0, \quad (54)$$

$\cdots$.

Solving of the above equations is direct and simple. The results read

$$a_N(t) = C_N, \quad (55)$$

$$a_{N-1}(t) = C_{N-1}, \quad (56)$$

$$a_{N-2}(t) = C_{N-2}, \quad (57)$$

$$a_{N-3}(t) = -[12C^2 N + 12CN(N-2) + 4N(N-1)(N-2)]C_N t + C_{N-3}, \quad (58)$$

$$a_{N-4}(t) = -[12C^2(N-1) + 12C(N-1)(N-3) + 4(N-1)(N-2)(N-3)]C_{N-1}t + C_{N-4}, \quad (59)$$

$$a_{N-5}(t) = -[12C^2(N-2) + 12C(N-2)(N-4) + 4(N-2)(N-3)(N-4)]C_{N-2}t + C_{N-5}, \quad (60)$$

$\cdots$.

Substituting $a_N(t)$, $a_{N-1}(t)$, $a_{N-2}(t)$, $a_{N-3}(t)$, $a_{N-4}(t)$ ... in (55–60) and $a_n(t)$ (for $n \geq N$) in (46), we obtain

$$\varphi_N(x,t) = C_N(x+x_0)^N + C_{N-1}(x+x_0)^{N-1} + C_{N-2}(x+x_0)^{N-2} + \{-[12C^2 N + 12CN(N-2) + 4N(N-1)(N-2)]C_N t + C_{N-3}\}(x+x_0)^{N-3} + \ldots. \quad (61)$$

Substituting $\varphi_N(x,t)$, $P_x = Q = -\frac{C}{(x+x_0)^2}$, $P = \frac{C}{x+x_0}$, $C = \frac{3\pm1}{2}$ into (5) and (6) and using $f(\varphi) = g(\varphi) = \ln(\varphi)$, we obtain rational solutions $H_N(x,t)$ and $G_N(x,t)$.

Because the terms reserved in the series solution (47) are arbitrary, we may obtain arbitrary many solutions similar to the one in (48).

Remark: We define $\varphi_N(x,t)$ as $\sum_{n=0}^{N} a_n(t)(x+x_0)^n$. According to my assumption that is $a_n(t) \equiv 0$ (for $n \geq N, N$ is an arbitrary constant), for an arbitrary positive number ε, and $n > N$, we can obtain

$$|r_n(x,t)| = |\varphi(x,t) - \varphi_N(x,t)| < \varepsilon,$$

so the convergence of the series for φ defined in (46) has been confirmed.

3. The Initial Value Problem and Exact Solutions of the Higher-order BK in (2+1) Dimensions

We suppose that the solution of (3) and (4) is of the form

$$H = f'\varphi_x + P(x,z,t), \tag{62}$$

$$G = g''\varphi_x\varphi_z + g'\varphi_{xz} + Q(x,z,t). \tag{63}$$

Through a discussion similar to the case of the higher-order BK system in (1+1) dimensions, we obtain

$$\varphi_t + 12P^2\varphi_x + 12(P\varphi_x)_x + 4\varphi_{xxx} = 0, \tag{64}$$

$$P_z = Q. \tag{65}$$

From (62–63) and $f(\varphi) = g(\varphi) = \ln(\varphi)$, we obtain a Backlund transformation of the BK equations in (2+1) dimensions:

$$H(x,z,t) = \partial \ln\varphi/\partial x + P, \tag{66}$$

$$G(x,z,t) = \partial^2 \ln\varphi/\partial x\partial z + Q, \tag{67}$$

where φ, P, and Q satisfy (64) and (65).

i) Making the ansatz $P(x,z,t) = P_0(z)$, $Q(x,z,t) = \partial P_0(z)/\partial z$, we obtain a transformation of the BK equations in (2+1) dimensions from (64–67):

$$H(x,z,t) = \partial \ln\varphi/\partial x + P_0(z), \tag{68}$$

$$G(x,z,t) = \partial^2 \ln\varphi/\partial x\partial z + P_0'(z). \tag{69}$$

ii) Assuming $P(x,z,t) = \varphi$, $Q(x,z,t) = \varphi_z$, we can derive another transformation of the BK equations in (2+1) dimensions from (64–67):

$$H(x,z,t) = \partial \ln\varphi/\partial x + \varphi, \tag{70}$$

$$G(x,z,t) = \partial^2 \ln\varphi/\partial x\partial z + \varphi_z. \tag{71}$$

iii) Similar to the case of the higher-order BK system in (1+1) dimensions, we can obtain three groups of exact solutions of the system (3) and (4) as follows:

$$H_1(x,z,t) = \frac{\sqrt{3}P_0(z)A\exp[36P_0^3(z)t]\cos[\sqrt{3}P_0(z)x]}{1+A\exp[36P_0^3(z)t]\sin[\sqrt{3}P_0(z)x]} + P_0(z), \tag{72}$$

$$[G_1(x,z,t) = \partial H_1(x,z,t)/\partial z, \tag{73}$$

$$H_2(x,z,t) = -\frac{\sqrt{3}P_0(z)A\exp[36P_0^3(z)t]\sin[\sqrt{3}P_0(z)x]}{1+A\exp[36P_0^3(z)t]\cos[\sqrt{3}P_0(z)x]} + P_0(z), \tag{74}$$

$$G_2(x,z,t) = \partial H_2(x,z,t)/\partial z, \tag{75}$$

$$H_3(x,z,t) = \tfrac{1}{2}\alpha\tanh\left\{\tfrac{1}{2}[\alpha x + (-4\alpha^3 - 12P_0(z)\alpha^2 - 12P_0^2(z)\alpha)t + \gamma]\right\} + \tfrac{1}{2}\alpha + P_0(z), \tag{76}$$

$$G_3(x,z,t) = \partial H_3(x,z,t)/\partial z. \tag{77}$$

Also a closed form of the solution for the initial value problem can be given:

$$H(x,z,t) = P_0(z) + \frac{\partial}{\partial x}\ln[\varphi(x,z,t)], \tag{78}$$

$$G(x,z,t) = \frac{\partial}{\partial z}H(x,z,t), \tag{79}$$

where

$$\varphi(x,z,t) = \frac{1}{2\pi(12t)^{\frac{1}{3}}}\int_{-\infty}^{\infty}\varphi(\zeta,0,0) \cdot \exp[(x-\zeta)P_0(z) - 20tP_0(z)^3] \cdot Ai\left(\frac{24P_0(z)^2t - (x-\zeta)}{(12t)^{\frac{1}{3}}}\right)\mathrm{d}\zeta \tag{80}$$

and $Ai(x)$ is an Airy function.

iv) Using $P = \frac{C}{x+P_0(z)}$, $P_z = Q$, and $C = \frac{3\pm1}{2}$, we obtain arbitrarily many solutions of (64) as follows:

$$\varphi_N(x,z,t) = C_N[x+P_0(z)]^N + C_{N-1}[x+P_0(z)]^{N-1} + C_{N-2}[x+P_0(z)]^{N-2} \tag{81}$$

$$+\{-[12C^2N+12CN(N-2)+4N(N-1)(N-2)]C_Nt + C_{N-3}\}[x+P_0(z)]^{N-3}+\ldots,$$

where N is an arbitrary positive integer. Substituting this $\varphi_N(x,z,t)$ into (62) and (63) and using $f(\varphi) = g(\varphi) = \ln(\varphi)$, we obtain the corresponding rational solutions $H_N(x,z,t)$, $G_N(x,z,t)$.

4. Conclusion

By using the extended homogeneous balance method, we obtain Backlund transformations and exact solutions, especially the closed form of the solution for the initial value problem of the higher order Broer-Kaup (BK) systems in (1+1) and (2+1) dimensions. The method used here is simple and can be generalized to deal with other classes of nonlinear equations.

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